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Sets of type-$(1, n)$ in biplanes☆

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Abstract

A set of type-(m, n) S is a set of points of a design with the property that each block of the design meets either m points or n points of S. If $m = 1$, S gives rise to a subdesign of the design. The parameters of sets of type-$(1, n)$ in finite projective planes were characterised by G. Tallini and M. Tallini Scafati with more generalised order condition. It follows from their result that, a set of type-$(1, n)$ exists only in the planes of square orders and it gives rise to either a Baer subplane or a unital of a finite projective plane of square order. In this paper, we characterise the parameters of sets of type-$(1, n)$ in biplanes of more extended order condition than prime power. It follows from the results that a set of type-$(1, n)$ in a biplane is either a Baer subdesign, a Hermitian subdesign or a subdesign with certain types of parameters. In addition, some examples of sets of type-$(1, n)$ are given in known biplanes.

Keywords: Blocking sets; Sets of type-$(1, n)$; Symmetric designs

1. Introduction

Let S be a subset of the point set of a 2–(v, k, λ) design D. For given integers m, n with $0 \le m < n \le k$, a set S is called a set of type-(m, n) in D, if each block of D meets S in either m points or n points. If a set of type-(m, n) is an s-set (of cardinality s), we refer to it as $(s; n, m)$*-set* in D. A block which meets S in i points is called an i*-secant*. A 1*-secant* is called a *tangent*. Denote by s the number of points of S, by t_j the number of j*-secants*, by b the number of blocks of D and by r the replication number of D which is the number of blocks passing through a point of D. Note that for each $j = m$ or n, we have $t_j \ge 1$

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(i.e. $t_j \neq 0$) directly from the definition. It is easy to verify [2] that the following linear equations hold:

$$\begin{aligned} &\text{(i)}\ t_m + t_n = b \\ &\text{(ii)}\ mt_m + nt_n = rs \\ &\text{(iii)}\ m(m-1)t_m + n(n-1)t_n = \lambda s(s-1). \end{aligned} \tag{1}$$

From the linear combination $mn \times$ (i) $-$ $(m+n-1) \times$ (ii) $+$ (iii) of the three equations in (1), we have the following Diophantine equation [2, 3, 5] as a necessary condition for the existence of sets of type-(n, m) in 2–(v, k, λ) design D:

$$\lambda s^2 - (r(m+n-1) + \lambda)s + bmn = 0 \tag{2}$$

where the discriminant $\Delta = (r(n+m-1) + \lambda)^2 + 4bmn\lambda$ of this quadratic equation with respect to s must be a non-negative square.

On the other hand, for a given $(s; m, n)$-set S in 2–(v, k, λ) design, let σ_j be the number of j*-secants* passing through a point P not in S, and let ρ_j be the number of j*-secants* passing through Q in S. Then we have the following properties of σ_j and ρ_j (see [5]).

Lemma 1. *Let S be an $(s; m, n)$-set in a given symmetric 2–(v, k, λ) design D. Then,*

(i) $\sigma_m + \sigma_n = k$ *and* $\rho_m + \rho_n = k$,
(ii) $m\sigma_m + n\sigma_n = \lambda s$ *and* $(m-1)\rho_m + (n-1)\rho_n = \lambda(s-1)$.

Note that (i) and (ii) imply that

$$\begin{aligned} \sigma_m &= \frac{nk - \lambda s}{n-m} & \rho_m &= \sigma_m - \frac{k-\lambda}{n-m} \\ \sigma_n &= \frac{\lambda s - mk}{n-m} & \rho_n &= \sigma_n + \frac{k-\lambda}{n-m}. \end{aligned} \tag{3}$$

It follows that for an $(s; m, n)$-set S in 2–(v, k, λ) design D there are constants $\sigma_m, \sigma_n, \rho_m$ and ρ_n such that for every point not in S there pass σ_m m-secants and σ_n n-secants, and for every point of S there pass ρ_m m-secants and ρ_n n-secants. From (3), since $\sigma_m, \sigma_n, \rho_m$ and ρ_n are integers, we have the following properties on divisibility, which can be seen in [2].

Lemma 2. *For any $(s; m, n)$-set in a given symmetric 2–(v, k, λ) design D, we have*

(i) $(n-m) \mid (k-\lambda)$
(ii) $(n-m) \mid \lambda(s-m)$.

An $(s; m, n)$-set S in a design D gives rise to a substructure of D with a set of blocks such that each block is defined as the set of points of S which are incident with a block of D and each block consists of more than one point. Particularly, a set of type-$(1, n)$ in a design D gives rise to a subdesign of D with block size n. An $(s; 1, n)$-set in a finite projective plane is sometimes called a blocking set. In 1966, Eq. (2) for blocking sets was given by Tallini Scafati [4], with the hypothesis of prime power order of a projective plane. By solving (2) when D is a projective plane of prime power order (i.e. in (2), put $\lambda = 1$, $b = v$ and $r - 1 = k - 1 = p^h$, p a prime, h a non-negative integer) and $m = 1$, she showed that $k - 1 = (n-1)^2$, which means an $(s; 1, n)$-set S exists only in the planes of square orders. In 1985, Tallini [3] generalised this result for $(s; 1, n)$-sets in finite projective planes of order $k - 1 = p^h(n - 1)$, where p is a prime and h is a non-negative integer, so that all possible sets of type-$(1, n)$ in the planes are completely characterised from the arithmetical point of view, stated in [3] as follows. Note that the order condition includes the cases of prime power orders and it can be assumed from Lemma 2 when $m = 1$ and $\lambda = 1$.

Result 3. Suppose S is an $(s; 1, n)$-set in a projective plane of order q and $q/(n-1) = p^h$ where p is a prime and h is a positive integer. Then $q = p^{2h}$, $n = 1 + \sqrt{q}$ and S is either a Baer subplane or a unital.

From the notation of Bose and Shrikhande [8], we define the following subdesigns of a symmetric 2–(v, k, λ) design D. A 2–(v^*, k^*, λ) subdesign of D is called a *Baer subdesign* of D if it is symmetric and $k^* = 1+\sqrt{k-\lambda}$. If $\lambda = 1$, it is a Baer subplane. A 2–(v^*, k^*, λ) subdesign of D is called a *Hermitian* subset of D if $v^* = (\sqrt{k-\lambda}/\lambda)(k-1) + 1$ and $k^* = 1 + \sqrt{k-\lambda}$. It is a unital (see [6]) when $\lambda = 1$. Note that an $(s; 1, n)$-set S gives rise to a 2–(s, n, λ) subdesign of D by taking n-secants as blocks of the subdesign. Thus, an $(s; 1, n)$-set will be called a Baer subdesign when $s = ((1 + \sqrt{k-\lambda})/\lambda)\sqrt{k-\lambda} + 1$, $n = 1+\sqrt{k-\lambda}$, and a Hermitian subset when $s = (\sqrt{k-\lambda}/\lambda)(k-1)+1$, $n = 1+\sqrt{k-\lambda}$, respectively. If a set of type-$(1, n)$ is a Baer subdesign, Eq. (2) provides another non-negative integral root which corresponds to a parameter set of another subdesign of parameters of Hermitian set (see [2]). Hence, Tallini's result (Result 3) shows that, if there is a set of type-$(1, n)$ in a finite projective plane of order q where $q/(n - 1)$ is a prime power, it is either a Baer subdesign or a Hermitian set, i.e. a Baer subplane or a unital in the projective plane, respectively.

In this paper, we pose the question whether the analogous result to Tallini and Tallini Scafati's holds in *biplanes* of order $k - 2$ which are symmetric 2–(v, k, λ) designs with $\lambda = 2$. From the divisibility properties of the parameters in Lemmas 2 and 4, we suppose a more generalised condition of order of biplane such as $(k-2)/(n-1)^2 = p^h$. Then, we find all possible non-negative integral solutions of Diophantine equation (2) for the case of biplanes so that all possible sets of type-$(1, n)$ are characterised from the arithmetical point of view. As a conclusion, these feasible solutions imply necessary conditions for existence of sets of type-$(1, n)$ as either Baer subdesigns, Hermitian sets or some subdesigns of biplanes with certain types of parameters as stated in Theorem 8. In the next section, some examples of sets of type-$(1, n)$ which exist in known biplanes are given.

2. Sets of type-$(1, n)$ in biplanes

We begin with a divisibility property of the parameters of an $(s; 1, n)$-set in a symmetric 2–(v, k, λ) design when $\lambda \geq 2$. Let S be an $(s; 1, n)$-set in a symmetric 2–(v, k, λ) design with $\lambda \geq 2$. In Eq. (2), if we put $m = 1$, then Eq. (2) is written as

$$\lambda^2 s^2 - \lambda(kn + \lambda)s + (k(k-1) + \lambda)n = 0 \tag{4}$$

by substituting $k(k-1)/\lambda+1$ for b and v. Let $k-\lambda=\alpha$, $n-1=\beta$, and $\lambda(s-1)=w$. Then (4) may be written as

$$w^2-(\alpha\beta+\alpha+\lambda\beta)w+\alpha(\beta+1)(\alpha+\lambda-1)=0. \tag{5}$$

From (5), we have the following lemma for the divisibility of the parameters.

Lemma 4. *Let S be an $(s;1,n)$-set of a symmetric 2–(v,k,λ) with $\lambda\geq 2$. Then, $(n-1)^2$ divides $(k-\lambda)(\lambda-1)$.*

Proof. We suppose with the notation as above, Eq. (5) can be stated as

$$w^2-(\alpha\beta+\alpha+\lambda\beta)w+\alpha(\alpha\beta+\beta(\lambda-1)+\alpha)=-\alpha(\lambda-1).$$

Since we supposed $\lambda\geq 2$ and Lemma 2 implies that β divides w and α, each term on the left hand side of this equation is divisible by β^2. Hence, the statement holds. □

Let $\mathcal{B}$ be a biplane of order $k-2$ (that is a symmetric 2–$(v,k,2)$ design). Let S be an $(s;1,n)$-set in $\mathcal{B}$. From now on, we use the notation $k-2=\alpha$, $n-1=\beta$ and $w=2(s-1)$. From (5), we have the following equation.

$$w^2-(\alpha\beta+\alpha+2\beta)w+\alpha(\alpha+1)(\beta+1)=0. \tag{6}$$

Recall $\beta^2 \mid \alpha$ from Lemma 4. Suppose $\alpha/\beta^2=p^h$ for a prime p and a non-negative integer h. Since $\beta \mid w$, if we divide (6) by β^2, and put $w_\beta=w/\beta$, we obtain the following quadratic.

$$w_\beta^2-\left(\alpha+\frac{\alpha}{\beta}+2\right)w_\beta+\frac{\alpha}{\beta^2}(\alpha+1)(\beta+1)=0$$

which may be written as

$$w_\beta^2-(\beta^2p^h+\beta p^h+2)w_\beta+p^h(\beta^2p^h+1)(\beta+1)=0. \tag{7}$$

In this section, we obtain every set of parameters satisfying Diophantine equation (7) so that we conclude, as stated in Theorem 8, that either $p^h=1$ or $p^h=2\beta-3$. Now we consider the following three cases, separately.

(I) $p\neq 2, \beta\geq 4$, (II) $p\neq 2, \beta<4$, (III) $p=2$.

We now begin with case (I).

Proposition 5. *Let S be an $(s;1,n)$-set in a biplane of order α. For a prime p and a non-negative h, suppose $\alpha/\beta^2=p^h$, $p\neq 2$ and $\beta\geq 4$. Then, either $\alpha=\beta^2$, or $\alpha=(2\beta-3)\beta^2$.*

Proof. If $h=0$, then $p^h=1$ and $\alpha=\beta^2$ as required. Suppose now that $h\geq 1$. Let x_1 and x_2 be two integer roots for (7). Then,

$$x_1+x_2=\beta^2p^h+\beta p^h+2 \tag{8}$$
$$x_1x_2=p^h(\beta^2p^h+1)(\beta+1). \tag{9}$$

Since $p \neq 2$, p divides x_1x_2 but p does not divide $x_1 + x_2$. Thus, one root, say x_1, and p are coprime and so we can put the other root $x_2 = cp^h$ for some positive integer c. If we substitute cp^h for x_2 and eliminate x_1 in (7), we have

$$p^h c^2 - (\beta^2 p^h + \beta p^h + 2)c + (\beta^2 p^h + 1)(\beta + 1) = 0 \tag{10}$$

which implies the following ratio:

$$p^h = \frac{2c - (\beta + 1)}{c^2 - (\beta^2 + \beta)c + \beta^2(\beta + 1)} \tag{11}$$

if the denominator is not 0.

Let $g_1(c)$ be the numerator of (11) and $g_2(c)$ be the denominator of (11). Then note that

(i) $g_2(\beta + 1) = \beta + 1 = g_2(\beta^2 - 1) > 0$,
(ii) $g_2(\beta + 2) = -\beta^2 + 2\beta + 4 = g_2(\beta^2 - 2) < 0$ if $\beta \geq 4$,
(iii) $g_1(c) = g_2(c)$ if and only if $c = \beta^2 + 1$ or $\beta + 1$.

Now with $\beta \geq 4$, it follows that $p^h < 1$ when $c < \beta + 1$ or $c > \beta^2 + 1$, and $p^h < 0$ when $\beta + 2 < c < \beta^2 - 2$; so, these cases cannot occur. Moreover, if $c = \beta + 1$ or $\beta^2 + 1$, then $p^h = 1$ and $\alpha = \beta^2$ as required.

Now, we evaluate ratio (11) for the remaining values of c, i.e. $c = \beta^2 - 1, \beta^2$. If $c = \beta^2$, by (11) $p^h = 2 - (\beta + 1)/\beta^2$ which is not an integer since $\beta > 1$. If $c = \beta^2 - 1$, by (11) we have

$$p^h = \frac{2\beta^2 - \beta - 3}{\beta + 1} = 2\beta - 3$$

and $\alpha = (2\beta - 3)\beta^2$ as required. $\square$

Now, we consider case (II).

Proposition 6. *Let S be an $(s; 1, n)$-set in a biplane of order α. For a prime p and a non-negative integer h, suppose $\alpha/\beta^2 = p^h$, $p \neq 2$ and $\beta < 4$. Then, either $\alpha = \beta^2$ or $\alpha = 27$, $\beta = 3$.*

Proof. If $h = 0$, then $p^h = 1$ and $\alpha = \beta^2$ as required. Suppose now that $h > 0$. As in the proof of the previous proposition, we observe Eq. (11) for the cases when $\beta = 1$, $\beta = 2$ and $\beta = 3$, respectively.

(i) $\beta = 1$; by (11) we have

$$p^h = \frac{2c - 2}{c^2 - 2c + 2}.$$

If $c = 1$ or $c > 2$, then $p^h < 1$ and we have a contradiction. If $c = 2$, then $p^h = 1$. Since $h \neq 0$, this case does not occur.

(ii) $\beta = 2$; by (11) we have

$$p^h = \frac{2c - 3}{c^2 - 6c + 12}.$$

Let D denote the denominator and N denote the numerator of the fraction. Note that $D = N$ if $c = 3, 5$, and that $D > N$ if $c < 3$ or $c > 5$, which means that $p^h < 1$. If $c = 4$, then $p^h = 5/4$ which is not an integer. Thus these cases do not occur. If $c = 3$ or 5, then $p^h = 1$. Since we suppose $h \neq 0$, this case does not occur.

(iii) $\beta = 3$; by (11) we have

$$p^h = \frac{2c - 4}{c^2 - 12c + 36}.$$

Let D' denote the denominator and N' denote the numerator of the fraction. Note that $D' = N'$ if $c = 4, 10$, and that $D' > N'$ if $c < 4$ or $c > 10$, which means $p^h < 1$. Thus we consider all integers c with $4 \leq c \leq 10$, as follows. If $c = 4$ or 10, then $p^h = 1$ so that $h = 0$; but we supposed $h \neq 0$. If $c = 5$, then $p^h = 6$. If $c = 6$ and $f(c)$ denotes the quadratic equation of (10) then $f(6) = 8 \neq 0$. If $c = 7$ then $p^h = 10$. If $c = 9$ then $p^h = 14/9$. Thus these cases do not occur. On the other hand, if $c = 8$ then $p^h = 12/4 = 3$ which implies $\alpha = 27$, $\beta = 3$. □

Notice that $\alpha = 27$ and $\beta = 3$ satisfy the condition $\alpha = (2\beta - 3)\beta^2$ stated in Proposition 5. Now we consider case (III).

Proposition 7. *Let S be an $(s; 1, n)$-set in a biplane of order α. For a non-negative integer h, suppose $\alpha/\beta^2 = 2^h$. Then, $\alpha = \beta^2$.*

Proof. If $h = 0$, then $2^h = 1$ and $\alpha = \beta^2$ as required. Suppose now that $h \geq 1$. With parameters $w_\beta = w/\beta = \lambda(s-1)/\beta = 2(s-1)/\beta$ and $p = 2$, Eq. (7) is stated as

$$w_\beta^2 - (2^h\beta^2 + 2^h\beta + 2)w_\beta + 2^h(2^h\beta^2 + 1)(\beta + 1) = 0 \tag{12}$$

which implies

$$w_\beta(2^h\beta^2 + 2^h\beta + 2 - w_\beta) = 2^h(2^h\beta^2 + 1)(\beta + 1). \tag{13}$$

Note that if $h \neq 0$, then $2 \mid w_\beta$ by (13). If we divide (13) by 2^2, it follows that

$$w_\beta'^2 - (2^{h-1}\beta^2 + 2^{h-1}\beta + 1)w_\beta' + 2^{h-2}(2^h\beta^2 + 1)(\beta + 1) = 0 \tag{14}$$

where $w_\beta' = w_\beta/2$. Let y_1 and y_2 be two roots of (14). Then

$$y_1 + y_2 = 2^{h-1}\beta^2 + 2^{h-1}\beta + 1 \tag{15}$$

$$y_1 y_2 = 2^{h-2}(2^h\beta^2 + 1)(\beta + 1). \tag{16}$$

Suppose $h \geq 3$. Since 2 divides $y_1 y_2$ and 2 does not divide $y_1 + y_2$, we can substitute $2^{h-2}t$ for one of the roots, say y_2, for some integer t. By substituting for y_2 and eliminating y_1 in (15) and (16) we obtain

$$2^{h-2}t^2 - (2^{h-1}\beta^2 + 2^{h-1}\beta + 1)t + (2^h\beta^2 + 1)(\beta + 1) = 0. \tag{17}$$

By multiplying (17) by 2^2, we have

$$2^h t^2 - (2^{h+1}\beta^2 + 2^{h+1}\beta + 4)t + 4(2^h\beta^2 + 1)(\beta + 1) = 0. \tag{18}$$

Then we obtain the following ratio;

$$2^h = \frac{4t - 4(\beta + 1)}{t^2 - (2\beta^2 + 2\beta)t + 4\beta^2(\beta + 1)}. \tag{19}$$

Let $r_1(t)$ be the numerator of (18) and $r_2(t)$ be the denominator of (18). Then we observe that

- $r_1(t) = r_2(t)$ if and only if $t = 2(\beta + 1)$ or $2(\beta^2 + 1)$,
- $r_2(2\beta^2 - 3) = -2\beta^2 + 6\beta + 9 = r_2(2\beta + 3) < 0$ if $\beta \geq 5$,
- $r_2(2\beta + 2) = 4(\beta + 1) = r_2(2\beta^2 - 2) > 0$.

If we suppose $\beta \geq 5$, note that $2^h < 0$ when $2\beta + 3 \leq t \leq 2\beta^2 - 3$ and that $2^h < 1$ when $t < 2(\beta + 1)$ or $t > 2(\beta^2 + 1)$; so, these cases do not occur. Now we consider the rest of the values of t, i.e. $t = 2\beta + 2$, $2\beta^2 - 2 \leq t \leq 2\beta^2 + 2$. The evaluations of (19) for these values of t are as follows:

(i) If $t = 2\beta^2 + 2$ or $2\beta + 2$ then $2^h = 1$ while we supposed $h \geq 3$.

(ii) If $t = 2\beta^2 + 1$ then $2^h = (8\beta^2 - 4\beta)/(6\beta^2 - 2\beta + 1) = (2\beta^2 - 2\beta - 1)/(6\beta^2 - 2\beta + 1) + 1 \notin \mathbb{Z}$ since $6\beta^2 - 2\beta - 1 > 2\beta^2 - 2\beta - 1$ for all integer $\beta \geq 1$.

(iii) If $t = 2\beta^2$ then $2^h = (8\beta^2 - 4\beta - 4)/(4\beta^2) = 2 - (\beta + 1)/(\beta^2) \notin \mathbb{Z}$, which implies a contradiction.

(iv) If $t = 2\beta^2 - 1$ then $2^h = (8\beta^2 - 4\beta - 8)/(2\beta^2 + 2\beta) = 4 - (6\beta + 4)/(\beta^2 + \beta) \notin \mathbb{Z}$, since $\beta^2 + \beta > 6\beta + 4$ if $\beta \geq 6$, and $2^h = 4 - 34/33 \notin \mathbb{Z}$ if $\beta = 5$.

(v) If $t = 2\beta^2 - 2$ then $2^h = (4(2\beta^2 - 2) - 4\beta - 4)/(4\beta + 4) = 2\beta - 3$, which is not possible since the right hand side is odd while the other side is even.

Hence if $\beta \geq 5$ and $h \geq 3$, no integral solutions exist.

Now, we suppose that $\beta < 5$ and $h \geq 3$. We consider the following cases.

(i) $\beta = 1$; from (19), we have

$$2^h = \frac{4t - 8}{t^2 - 4t + 8}.$$

If $t = 4$, then $2^h = 1$, i.e. $0 = h \ngeq 3$, a contradiction; otherwise, $t^2 - 4t + 8 > 4t - 8$; a contradiction. Thus, these cases do not occur.

(ii) $\beta = 2$; from (19), we have

$$2^h = \frac{4t - 12}{t^2 - 12t + 48}.$$

If $t \leq 5$ or $t \geq 11$ then $2^h < 1$. When $t = 6$ or 10, we have $2^h = 1$; but $h \ngeq 3$. If $t = 7, 8$ and 9, we have $2^h = 16/13, 16/13$ and $24/21 \notin \mathbb{Z}$, respectively. Thus, these cases do not occur.

(iii) $\beta = 3$; from (19), we have

$$2^h = \frac{4t - 16}{t^2 - 24t + 144}.$$

We evaluate this ratio for every possible value of t as follows.

If $t = 8$ or 20, then $2^h = 1$ but $h \ngeq 3$. If $t < 8$ or $t > 20$ then $2^h < 1$. If $t = 9$ then $2^h = 20/9 \notin \mathbb{Z}$. If $t = 10$ then $2^h = 24/4 = 6$. If $t = 11$ then $2^h = 28/1$. If $t = 12$ then the denominator of the ratio is 0; thus, we have $t = 4$ from (17), which contradicts that $t = 12$. If $t = 13$ then $2^h = 36/1$. If $t = 14$ then $2^h = 40/4 = 10$. If $t = 15$ then $2^h = 44/9 \notin \mathbb{Z}$. If $t = 16$ then $2^h = 48/16 = 3$. If $t = 17$ then $2^h = 52/25 \notin \mathbb{Z}$. If $t = 18$ then $2^h = 56/36 \notin \mathbb{Z}$. If $t = 19$ then $2^h = 60/49 \notin \mathbb{Z}$.

Thus, these cases do not occur.

(iv) $\beta = 4$; from (19), we have

$$2^h = \frac{4t - 20}{t^2 - 40t + 320}.$$

We evaluate this ratio for every possible value of t as follows.

If $t = 10$ or 34, then $2^h = 1$; but $h \ngeq 3$. If $t < 10$ or $t > 34$ then $2^h < 1$. If $12 \leq t \leq 28$ then $2^h < 0$, since the roots of the denominator are $20 \pm \sqrt{80}$. If $t = 11$ then $2^h = 24/1$. If $t = 29$ then $2^h = 96/1$. If $t = 30$ then $2^h = 100/20 = 5$. If $t = 31$ then $2^h = 104/41 \notin \mathbb{Z}$. If $t = 32$ then $2^h = 108/64 \notin \mathbb{Z}$. If $t = 33$ then $2^h = 112/89 \notin \mathbb{Z}$.

Thus, these cases do not occur.

Hence, when $p = 2$ and $h \geq 3$, for all positive integers β, (17) has no integral solution t.

Now, we suppose $h < 3$. If $h = 0$, then $\alpha = \beta^2$ as required. Suppose $h = 1$. Then, (13) is written as

$$w_\beta^2 - (2\beta^2 + 2\beta + 2)w_\beta + 2(2\beta^2 + 1)(\beta + 1) = 0 \tag{20}$$

which implies $2 \mid w_\beta$. If we divide (20) by 2, we have

$$2t^2 - 2(\beta^2 + \beta + 1)t + (2\beta^2 + 1)(\beta + 1) = 0 \tag{21}$$

where $t = w_\beta/2$, which implies the following ratio

$$2 = \frac{-(\beta + 1)}{t^2 - (\beta^2 + \beta + 1)t + \beta^2(\beta + 1)} = \frac{-(\beta + 1)}{(t - (\beta + 1))(t - \beta^2)}. \tag{22}$$

Note that t is an integer with $\beta + 1 < t < \beta^2$ since all parameters are positive integers and the ratio is 2. If $g_2(t)$ denotes the denominator of (22), $|g_2(\beta + 2)|$ is the minimal possible integer value of $|g_2(t)|$ in this range since $|g_2(t)|$ is increasing up to the turning point $t = (\beta^2 + \beta + 1)/2$ of $g_2(t)$. Thus,

$$2 = \frac{-(\beta + 1)}{g_2(t)} \leq \frac{\beta + 1}{\beta^2 - \beta - 2} = \frac{1}{\beta - 2}.$$

Hence, if $\beta = 1$ or $\beta > 2$, we have a contradiction. If $\beta = 2$, (21) implies that

$$2t^2 - 14t + 27 = 0$$

which does not have any real root. We conclude that there is no integer solution of (20). Finally, we consider the case that $h = 2$. From (13),

$$w_\beta^2 - (4\beta^2 + 4\beta + 2)w_\beta + 4(2\beta^2 + 1)(\beta + 1) = 0. \tag{23}$$

By (23), since $2 \mid w_\beta$, we obtain the following

$$f(t) = t^2 - (2\beta^2 + 2\beta + 1)t + (4\beta^2 + 1)(\beta + 1) = 0 \tag{24}$$

where $t = w_\beta/2$. Note that

$$\begin{aligned} &f(2\beta + 2) = 3\beta + 3 > 0 \qquad \text{for all } \beta \geq 1, \qquad \text{and} \\ &f(2\beta + 3) = -(\beta + 1)(2\beta - 7) < 0 \qquad \text{if } \beta \geq 4. \end{aligned}$$

Hence if $\beta \geq 4$, (24) has a non-integral solution t_0 such that $2\beta + 2 < t_0 < 2\beta + 3$. This means that there is no integral solution of (24) since the leading coefficient of (24) is 1 and the coefficients are all integers.

For each $\beta = 1, 2, 3$, from Eq. (24) we have $t^2 - 5t + 10 = 0$, $t^2 - 13t + 51 = 0$, $t^2 - 25t + 148 = 0$, respectively. Each quadratic does not have any integral solution. Thus no cases arise when $h = 2$. It follows that the proposition holds. □

From the propositions, we conclude the following theorem.

Theorem 8. *Let S be an $(s; 1, n)$-set in a biplane of order $k-2$ and $(k-2)/(n-1)^2 = p^h$, p a prime and h is a non-negative integer. Then, either $k - 2 = (n - 1)^2$, or $k - 2 = (2n - 5)(n - 1)^2$.*

Note that if we suppose $p^h = 1$ then (7) has two integer solutions $w_\beta = \beta^2 + 1$ and $w_\beta = \beta + 1$ such that one implies that S is a Hermitian set and the other is a Baer subdesign. Moreover, if we suppose $p^h = 2\beta - 3$, then (7) has two integer solutions $w_\beta = (\beta - 1)(\beta + 1)(2\beta - 3)$ and $w_\beta = (\beta - 1)(2\beta + 1)$ by substitution and factorisation of (7).

3. Examples in known biplanes

In this section, we deal with some examples of sets of type-$(1, n)$ in known biplanes. The existences of biplanes with parameters 2–$(v, k, 2)$ satisfying the Bruck–Ryser–Chowla theorem (see [1]) are known only when $k = 4, 5, 6, 9, 11, 13$ (see [7]). Among these biplanes, Theorem 8 implies that there are two biplanes of order $k - 2 = 4$ or 9 which possibly have a set of type-$(1, n)$. We observe sets of type-$(1, n)$ in these biplanes of order 6 and 11, respectively, as follows.

Example 9. There may be two types of $(s; 1, n)$-sets S in a biplane of order 4 (which is a symmetric 2–$(16, 6, 2)$ design); one is a $(4; 1, 3)$-set which implies a 2–$(4, 3, 2)$ Baer subdesign, and the other is a (6; 1, 3)-set which implies a 2–$(6, 3, 2)$ Hermitian subset.

$$B_1 = \begin{pmatrix} J-I & I & I & I \\ I & J-I & I & I \\ I & I & J-I & I \\ I & I & I & J-I \end{pmatrix} = \begin{pmatrix} 0111100010001000 \\ 1011010001000100 \\ 1101001000100010 \\ 1110000100010001 \\ 1000011110001000 \\ 0100101101000100 \\ 0010110100100010 \\ 0001111000010001 \\ 1000100001111000 \\ 0100010001110100 \\ 0010001010110010 \\ 0001000111000001 \\ 1000100010000111 \\ 0100010001001011 \\ 0010001000101101 \\ 0001000100011110 \end{pmatrix}$$

Fig. 1. Biplane B_1.

It is known that there are three non-isomorphic biplanes of order 4 (see [7]) with the following incidence matrices. We show that each biplane has a 2–(4, 3, 2) Baer subdesign and a 2–(6, 3, 2) Hermitian subset.

Let

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that I is the identity 4×4 matrix, J is the 4×4 matrix with all entries 1 and P is a cyclic permutation matrix of order 3. Then we have the following three non-isomorphic biplanes B_1, B_2 and B_3 of order 4, represented as incidence matrices in Figs. 1–3, respectively.

For each incidence matrix, let the points of the corresponding biplane be written as $P_1, P_2, \ldots, P_{16}$ with respect to the order of rows and let the blocks be denoted by $l_1, l_2, \ldots, l_{16}$ with respect to the order of columns. Then, for $i = 1, 2, 3$, the subset of points $S_i = \{P_1, P_2, P_3, P_4\}$ is a (4; 1, 3)-set of B_i and the associated Baer subdesign of B_i from S_i is

$$S_i = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

On the other hand, for instance, if we take $S_1' = \{P_2, P_3, P_4, P_6, P_{11}, P_{16}\}$, it is a

$$
B_2 = \begin{pmatrix} J-I & I & I & I \\ I & J-I & I & I \\ I & I & (J-I)P & IP \\ I & I & IP & (J-I)P \end{pmatrix} = \begin{pmatrix} 0111100010001000 \\ 1011010001000100 \\ 1101001000100010 \\ 1110000100010001 \\ 1000011110001000 \\ 0100101101000100 \\ 0010110100100010 \\ 0001111000010001 \\ 1000100001111000 \\ 0100010011100001 \\ 0010001010110100 \\ 0001000111010010 \\ 1000100010000111 \\ 0100010000011110 \\ 0010001001001011 \\ 0001000100101101 \end{pmatrix}
$$

Fig. 2. Biplane B_2.

$$
B_3 = \begin{pmatrix} J-I & I & I & I \\ I & J-I & I & I \\ I & I & (J-I)P^2 & I \\ I & I & I & (J-I)P^2 \end{pmatrix} = \begin{pmatrix} 0111100010001000 \\ 1011010001000100 \\ 1101001000100010 \\ 1110000100010001 \\ 1000011110001000 \\ 0100101101000100 \\ 0010110100100010 \\ 0001111000010001 \\ 1000100001111000 \\ 0100010011010010 \\ 0010001011100001 \\ 0001000110110100 \\ 1000100010000111 \\ 0100010000101101 \\ 0010001000011110 \\ 0001000101001011 \end{pmatrix}
$$

Fig. 3. Biplane B_3.

(6; 1, 3)-set in B_1 which implies a Hermitian subset of B_1 represented as the following incidence matrix:

$$
S_1' = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.
$$

Example 10. We consider a construction of a biplane of order 9 (i.e. a symmetric 2–(56, 11, 2) design) which is expressed in Cameron ([9, p. 88]) as follows.

Let Ω be a set of 10 points carrying the Möbius plane M of order 3 which is a 3–(10, 4, 1) design (see [1, p. 134]). If $P_1, P_2 \in \Omega$, the set $\Omega - \{P_1, P_2\}$ can be uniquely expressed as the union of two disjoint blocks t_1, t_2 of M. Each of the four blocks incident with P_1 and P_2 meets one of t_1, t_2 in two points; so each of t_1, t_2 is partitioned into two sets of two points each. These two sets are the diagonals of a unique quadrangle on the points of the block. On $(b) = \{P_0\} \cup \Omega$, define a set $[b]$ of graphs as follows: for each $\{P_1, P_2\} \subset \Omega$, the disjoint union of the triangle $\{P_0, P_1, P_2\}$ and the two quadrangles constructed on the disjoint blocks complementing $\{P_1, P_2\}$, is in $[b]$; and these are all the members of $[b]$. Then $[b]$ implies Hussain graphs and so it defines a biplane C which is of symmetric 2–(56, 11, 2) design. In this construction, due to Cameron, let t_1 be partitioned into $\{Q_1, Q_2\}$ and $\{R_1, R_2\}$ such that $\{P_1, P_2, Q_1, Q_2\}$ is the block disjoint from t_2 and not containing $\{R_1, R_2\}$. Similarly, $\{P_1, P_2, R_1, R_2\}$ is the block disjoint from t_2 and not containing $\{Q_1, Q_2\}$. Then, we define a point set **P** and a block set **B** as follows:

P is the union of points of t_2 and three Hussain graphs determined by $\{P_1, P_2\}$, $\{Q_1, Q_2\}$, $\{R_1, R_2\}$;

B is the set of six edges of 4 points of t_2.

Then, these **P** and **B** give rise to a 2–(7, 4, 2) subdesign of C which is a Baer subdesign of C.

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